



Stream function approach for determining optimal surface currents

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Received 8 January 2003; received in revised form 3 June 2003; accepted 3 June 2003

Abstract

In many areas in industrial engineering, one may be faced with the question how an electromagnetic device has to be designed such that both a rather complex set of requirements such as geometrical constraints has to be fulfilled, and of which the magnetic properties has to be optimal in some sense. Given an electromagnetic design, a variety of methods exist to compute the additional magnetic properties and hence verify the constraints. However, the problem in which the optimal parameters are to be calculated given a set of constraints, is in general harder to solve. In this paper, we focus on quasi-static electromagnetic problems, where the problem is to find a certain conductor shape confined to an arbitrary but given surface, and electromagnetic properties are prescribed. Also conductive surfaces may be present, which affect these electromagnetic properties. With some additional assumptions the shape optimization problem can be formulated as a quadratic optimization problem with linear constraints.

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Keywords: Stream function; Surface currents; Topological optimization

1. Introduction

This paper discusses an approach to solving certain types of electromagnetic problems, in particular those occurring in the design of electromagnetic or electromechanical devices. In these situations often an electric conductor must be given a certain shape such that a number of electrical and/or magnetic properties are optimal. Furthermore, constraints may be imposed on a number of properties, for example geometric properties (such as maximum wire length, or all conductors contained within a certain volume), magnetic properties (such as prescribed magnetic field) or electric properties (such as self-inductance and resistance). Examples are the design of multi-poles used in particle accelerators and gradient coils for magnetic resonance imaging devices, where the spatial distribution of the magnetic field is prescribed, and the resistance and/or the self-inductance has to be minimal. These type of problems are often denoted as *field synthesis* or *shape optimization* problems.

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General approaches are described in [4,5]; an overview of recent open problems can and/or be found in [7]. Dependent of the specific characteristics of the problem, a number of dedicated approaches have been developed. Recent examples include the design of antennas [10], gradient coils for Magnetic Resonance Imaging (MRI) [16] and magnets for MRI [9]. These examples apply the stream function indirectly to get the conductor shape, after determining the (surface) current density, from which the stream function is constructed. In this paper we show that, following the approach of [14], it is more advantageous to model the stream function directly.

Given a conductor shape such as in Fig. 1, and hence a known current distribution and material properties, resulting electromagnetic properties can be computed. For simple geometries analytical expressions may be available, whereas for more complicated problems numerical methods have to be used, such as Finite Element methods or Boundary Element methods [3].

In this paper, we refer to the shape optimization problem as the problem where the optimal degrees of freedom are to be determined from a given set of constraints on the electromagnetic field, as opposed to the problem where this electromagnetic field has to be computed from a given set of parameters. Note that the shape optimization problem requires us to be able to handle at least the latter problem.

An often successful approach for handling the shape optimization problem consists of parameterizing the problem, and constructing constraints for the parameters representing the physical constraints and an objective function representing the required criterion for optimization. In the example in Fig. 1, the parameter space could be a finite subspace of the collection of all possible conductor shapes. In general this leads to a nonlinear optimization problem. The drawback is then that, in general, physical understanding of the problem is needed to find a good initial set of parameters, because convergence of the optimization problem to the global optimum cannot be guaranteed in general. As a consequence, when confronted with failing convergence, one may not be sure whether the lack of convergence was due to wrong parameters for the optimization algorithm (such as the initial guess), or that the physical problem in combination with the constraints indeed cannot be solved.

Another approach, which is the approach we will adopt in this paper, is to drop the restriction that the space of admissible solutions is (a subspace of) the space of possible conductor shapes. Instead we will use the more general space of possible current distributions. This approach can be applied if the geometrical constraints can be translated into the restriction that the currents are restricted to be within a certain prescribed volume. The advantage of this approach is that the collection of possible current distributions in a volume is much simpler to approximate by a finite set of parameters. Furthermore, if the constraints are linear in the parameters and the objective function is linear or quadratic with a positive-definite Hessian, globally convergent and robust optimization methods which find the global optimum [8,12] can be used. In

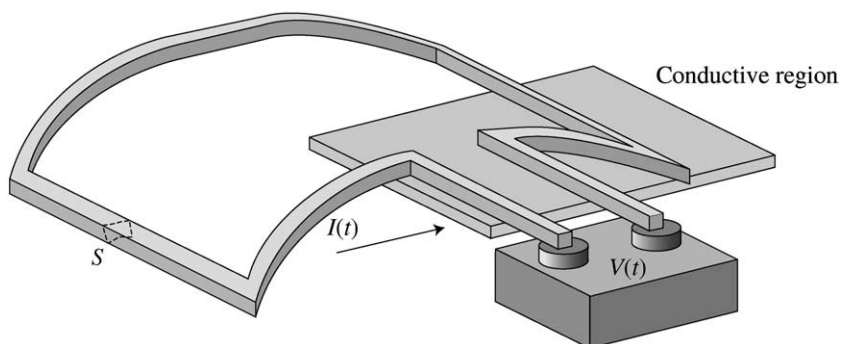


Fig. 1. Example of a conductor shape.

this case, failing convergence indeed indicates a conflict on the constraints, signalling the user that there is no optimal solution.

A drawback is that the conductor shape is not found directly. This shape has to be derived from the current distribution, a process which we denote as converting to a conductor. In this conversion process, the electromagnetic properties are slightly modified, and it is the aim of the conversion to keep this change minimal. In this paper, the stream function, which is a representation of the surface current distribution, is used which makes this conversion both simple and accurate.

The stream function approach as a method of shape optimization has aspects which are similar to the level set method [15] and the homogenization method [1]. In the level set method, the solution is represented as the level set of a function $\phi(\mathbf{x})$, resembling the use of the stream function. The homogenization method replaces a complex and highly localized property by an equivalent ‘homogenized’ field, reducing complexity considerably and often retaining linearity of derived properties. This resembles the replacement of a complex conductor shape by the current density. Both methods have recently been applied successfully to a number of topological optimization problems, see for example [13].

2. Problem description

This section gives an overview of the general problem under discussion. In subsequent sections, we will introduce simplifications which enable us to solve the problem efficiently.

The electric current is described by the current density, which is a vector field representing the velocity of free electric charges (e.g., electrons). The current density is a time and spatially dependent vector field, and will be denoted by $\mathbf{J}(\mathbf{x}, t)$. Furthermore, two disjoint volumes V_{source} and V_{ind} are defined. Denote the union of V_{source} and V_{ind} as V , then we require that

$$\mathbf{x} \notin V := V_{\text{source}} \cup V_{\text{ind}} \Rightarrow \mathbf{J}(\mathbf{x}, t) = \mathbf{0}, \quad (1)$$

where V_{source} represents the region where currents are flowing primarily because they are driven by a voltage source. V_{ind} on the other hand represents regions which are not connected to a power source, but currents (eddy currents) may be flowing here as a result Faraday’s induction law. Note that both V_{source} and V_{ind} may consist of a finite set of mutually disjoint volumes.

We now state our objective, at first phrased in very general terms:

Problem 1 (General). Determine

$$\mathbf{J}(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in V_{\text{source}},$$

where a number of constraints may be set on $\mathbf{J}(\mathbf{x}, t)$ in terms of the resulting electromagnetic properties.

In order to derive a robust method for this problem, we will introduce simplifications. The physical model fulfilling these simplifications will be discussed in the following section.

First recall that after determining the current density $\mathbf{J}(\mathbf{x}, t)$ the conductor shape has to be derived (conductor conversion). The strategy we choose is to require that $\mathbf{J}(\mathbf{x}, t)$ for $\mathbf{x} \in V_{\text{source}}$ can be written as

$$\mathbf{J}(\mathbf{x}, t) = I(t)\tilde{\mathbf{J}}_{\text{source}}(\mathbf{x}), \quad \mathbf{x} \in V_{\text{source}}, \quad (2)$$

in which case the conductor conversion can then be based on the static field $\tilde{\mathbf{J}}_{\text{source}}(\mathbf{x})$.

Given a $\tilde{\mathbf{J}}_{\text{source}}(\mathbf{x})$, $\mathbf{x} \in V_{\text{source}}$, and some ‘test function’ $I(t)$, the current density $\mathbf{J}(\mathbf{x}, t)$ is known both in V_{source} and V_{ind} , and hence all related electromagnetic properties. To allow applying linear or quadratic optimization methods, we require that the constraints on the electromagnetic properties are linear, and the

objective function linear or quadratic in $\tilde{\mathbf{J}}_{\text{source}}(\mathbf{x})$, and hence in $I(t)$. Note that minimizing inductance or resistance is equivalent to minimizing magnetic energy and dissipation, respectively; both are quadratic functions in $\mathbf{J}(\mathbf{x}, t)$. Therefore a quadratic objective functions seems to be a natural choice. Finally, we assume that V_{source} and V_{ind} are ‘thin’, and can therefore be approximated by surfaces.

This results in the following:

Problem 2. Determine

$$\tilde{\mathbf{J}}_{\text{source}}(\mathbf{x}) \quad \text{for } \mathbf{x} \in V_{\text{source}},$$

where $\tilde{\mathbf{J}}_{\text{source}}(\mathbf{x})$ is the solution of an optimization problem with a linear or quadratic objective function, with linear constraint functions. The media are static, linear and isotropic, and V_{source} and V_{ind} are thin.

Note that the problem may be generalized by stating that more than one conductor shape (each constrained within a certain volume $V_{\text{source}}^{(i)}$) must be determined, that is

$$\text{Determine } \tilde{\mathbf{J}}_{\text{source}}^{(i)}(\mathbf{x}), \quad \mathbf{x} \in V_{\text{source}}^{(i)}, \quad i = 1, \dots, N,$$

where N is the number of conductor shapes to be determined. If the volumes are mutually disjoint, this is a trivial extension of the method presented in this paper, and will therefore not be discussed further. Hence we may assume only one conductor path will have to be determined.

3. Physical model

As stated in the previous section, we first consider a model involving volume currents denoted by $\mathbf{J}(\mathbf{x}, t)$. In subsequent sections, we will restrict the volume currents to be surface currents, since then a scalar representation of the surface currents can be used, the stream function. It will be shown that the stream function has properties which makes it very useful for numerical applications. However, the basic simplifications of the model and the resulting properties can also be derived for the more general case.

Since the media are static, we can use Maxwell’s equations for stationary media (see for example [11]), which gives the relations between the magnetic flux density $\mathbf{B}(\mathbf{x}, t)$, the magnetic field strength $\mathbf{H}(\mathbf{x}, t)$, the electric flux density $\mathbf{D}(\mathbf{x}, t)$, the electric field strength $\mathbf{E}(\mathbf{x}, t)$, the current density (motion of free electric charges) $\mathbf{J}(\mathbf{x}, t)$ and the electric charge density $\rho(\mathbf{x}, t)$. Since we only a problem with good conductors we assume that $\rho(\mathbf{x}, t) = 0$.

The first simplification comes from the observation that for linear and isotropic media Maxwell’s equations relations become linear as well. This means that in this case the following constitutive relations exist:

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{x} \in \mathbb{R}^3, \quad \epsilon(\mathbf{x}) \text{ is the electric permittivity,} \quad (3)$$

$$\mathbf{B} = \mu \mathbf{H}, \quad \mathbf{x} \in \mathbb{R}^3, \quad \mu(\mathbf{x}) \text{ is the magnetic permeability,} \quad (4)$$

$$\mathbf{J} = \sigma \mathbf{E}, \quad \mathbf{x} \in V_{\text{ind}}, \quad \sigma(\mathbf{x}) \text{ is the conductivity.} \quad (5)$$

Note that (5) holds only for $\mathbf{x} \in V_{\text{ind}}$, because of (1) and that for $\mathbf{x} \in V_{\text{source}}$ the current density is to be determined.

Therefore $\mathbf{J}(\mathbf{x}, t)$, $\mathbf{H}(\mathbf{x}, t)$ and $\mathbf{E}(\mathbf{x}, t)$ can be used to describe all relevant vector fields, with the following differential relations (Maxwell):

$$\nabla \cdot (\epsilon \mathbf{E}) = 0, \quad (6)$$

$$\nabla \cdot (\mu \mathbf{H}) = 0, \quad (7)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad (8)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t}. \quad (9)$$

Note that here the equations are written in differential form; the equivalent integral formulations are to be used to analyze situations where the material properties, and hence the fields, are not continuous. Also, from (6) and (9) the following necessary condition for $\mathbf{J}(\mathbf{x}, t)$ follows:

$$\nabla \cdot \mathbf{J} = 0 \quad \text{in } \mathbb{R}^3. \quad (10)$$

Let $H(\text{div}; V)$ be the Hilbert space of vector functions defined on V of which the length and divergence are Lebesgue measurable and square-integrable, with inner product (see also [6])

$$(\mathbf{f}, \mathbf{g}) = \int_V \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) dV + \int_V (\nabla \cdot \mathbf{f})(\nabla \cdot \mathbf{g}) dV, \quad \mathbf{f}, \mathbf{g} \in H(\text{div}; V).$$

Denote by $N^0(\text{div}; V)$ the linear subspace of $H(\text{div}; V)$ consisting of functions with zero divergence, then according to (10), $\mathbf{J} \in N^0(\text{div}; V)$. Because V is compact, $N^0(\text{div}; V)$ is separable, so there exists a countable set of basis functions $(\hat{\mathbf{J}}_n(\mathbf{x}))_{n \in \mathbb{N}}$ such that every $\mathbf{J}(\mathbf{x}, t) \in N^0(\text{div}; V)$ can be written as

$$\mathbf{J}(\mathbf{x}, t) = \sum_{n=1}^{\infty} I_n(t) \hat{\mathbf{J}}_n(\mathbf{x}).$$

Without loss of generality we may assume that each basis function $\hat{\mathbf{J}}_n(\mathbf{x})$ has its support either in V_{source} or in V_{ind} , thus separating the indices in the sets $\mathbb{N}_{\text{source}}$ and $\mathbb{N}_{\text{ind}} := \mathbb{N} \setminus \mathbb{N}_{\text{source}}$, respectively. In Appendix A, it is shown that for the quasi-static case there exist numbers M_{mn} and R_{mn} such that the relation between currents in V_{source} and V_{ind} is expressed by

$$\sum_{m=1}^{\infty} \left[M_{mn} \frac{dI_m(t)}{dt} + R_{mn} I_m(t) \right] = 0, \quad n \in \mathbb{N}_{\text{ind}}, \quad (11)$$

where M_{mn} is called the mutual inductance between basis functions $\hat{\mathbf{J}}_m$ and $\hat{\mathbf{J}}_n$, and R_{mn} can analogously be given the term mutual resistance. Problems 1 and 2 state that the source currents are to be determined; since they are given by $\mathbf{J}(\mathbf{x}, t) = \sum_{n \in \mathbb{N}_{\text{source}}} I_n(t) \hat{\mathbf{J}}_n(\mathbf{x})$ for $\mathbf{x} \in V_{\text{source}}$, the degrees of freedom are $(I_n(t))_{n \in \mathbb{N}_{\text{source}}}$. Eq. (11) defines – together with suitable boundary or periodicity conditions – the induction currents, given by $\mathbf{J}(\mathbf{x}, t) = \sum_{n \in \mathbb{N}_{\text{ind}}} I_n(t) \hat{\mathbf{J}}_n(\mathbf{x})$ for $\mathbf{x} \in V_{\text{ind}}$.

4. Surfaces and stream functions

The relations presented in the previous sections hold for general geometries. In practice, a situation often occurs where the geometries have at least one dimension which is small compared to the region of interest. In these situations, we can represent the current densities by surface currents.

From now on we shall assume that the regions V_{source} and V_{ind} are ‘thin’ and can be described by a finite set of simply connected, orientable, compact and piecewise smooth surfaces S_k , $k = 1, \dots, K$, in

combination with a ‘thickness’ function $d : \bigcup_{k=1}^K S_k \rightarrow \mathbb{R}^+$. The boundaries (if any) are assumed to consist of a finite number of piecewise continuous simple closed curves $C_l, l = 1, \dots, L$. The collection of all surfaces S_k is denoted by $S := \bigcup_{k=1}^K S_k$.

The surface current density is denoted by $\mathbf{j}(\mathbf{x}), \mathbf{x} \in S$. Note that we drop the explicit dependency of time from now on. Expressions given as volume integrals with volume current density $\mathbf{J}(\mathbf{x})$ can be translated into surface integrals involving $\mathbf{j}(\mathbf{x})$ by substituting $\mathbf{j}(\mathbf{x}) = d(\mathbf{x})\mathbf{J}(\mathbf{x})$ and $dS = dV/d(\mathbf{x})$.

Recall that the surface divergence of \mathbf{j} , denoted by $\nabla_S \cdot \mathbf{j}$, is defined as

$$\nabla_S \cdot \mathbf{j}(\mathbf{x}) := \lim_{\substack{|S'| \rightarrow 0 \\ S' \subset S \\ \mathbf{x} \in S'}} \frac{1}{|S'|} \int_{\partial S'} (\mathbf{n}(\mathbf{x}) \times \mathbf{j}(\mathbf{x})) \cdot d\mathbf{l}, \quad \mathbf{j} : S \rightarrow \mathbb{R}^3,$$

with $|S'|$ denoting the area of S' . Here $\mathbf{n}(\mathbf{x})$ denotes the normal in \mathbf{x} . From this point, we will write \mathbf{n} instead of $\mathbf{n}(\mathbf{x})$, assuming implicit dependency on \mathbf{x} .

For a surface current $\mathbf{j}(\mathbf{x})$ property (10) therefore translates into

$$\nabla_S \cdot \mathbf{j} = 0,$$

which is equivalent to requiring that

$$\phi_C := \int_C \mathbf{j}(\mathbf{x}) \cdot (d\mathbf{l} \times \mathbf{n}) = \int_C (\mathbf{n} \times \mathbf{j}(\mathbf{x})) \cdot d\mathbf{l} = 0 \quad \text{for any closed contour } C \subset S.$$

See also Fig. 2(a) for a graphical interpretation. Note that ϕ_C is the current through C , which must be zero for a closed contour.

This property can be used to introduce a scalar field $\psi(\mathbf{x})$.

Definition 3. For every surface S_k , choose a fixed reference point \mathbf{a}_k . Then the *stream function* $\psi(\mathbf{x})$ corresponding to the surface current density $\mathbf{j}(\mathbf{x})$ is

$$\psi(\mathbf{x}) := \int_{\mathbf{a}_k}^{\mathbf{x}} (\mathbf{n} \times \mathbf{j}(\mathbf{x})) \cdot d\mathbf{l}, \quad \mathbf{x} \in S_k, \tag{12}$$

where the path from \mathbf{a}_k to \mathbf{x} must lie completely in S_k (see Fig. 2(b)).

Hence $\psi(\mathbf{x})$ is the current through a line on S_k from \mathbf{a}_k to \mathbf{x} . Due to (12) $\psi(\mathbf{x})$ does not depend on the path chosen from \mathbf{a}_k to \mathbf{x} .

If \mathbf{x} is not at the boundary of S , then $\mathbf{j}(\mathbf{x})$ is given from $\psi(\mathbf{x})$ by

$$\mathbf{j} = \nabla\psi \times \mathbf{n}. \tag{13}$$

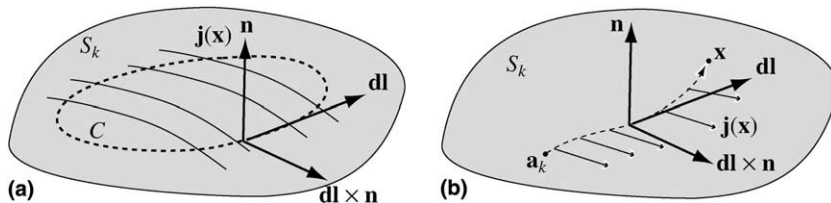


Fig. 2. The integration path and vectors used in the definition of the stream function.

Applying this inverse formula would formally require that $\psi(\mathbf{x})$ is defined outside S . However, only tangential surface derivatives are needed. For example, if S_k is described as a parameterized mapping from a region $U \subset \mathbb{R}^2$ to \mathbb{R}^3 :

$$S_k : \{\mathbf{x} = \mathbf{x}(u, v) \mid (u, v) \in U\},$$

then

$$\mathbf{j}(\mathbf{x}) = \left(\frac{\partial \psi}{\partial v} \frac{\partial \mathbf{x}}{\partial u} - \frac{\partial \psi}{\partial u} \frac{\partial \mathbf{x}}{\partial v} \right) / \left(\left\| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right\| \right).$$

Another very useful property which follows from its definition is

Corollary 4. *The curves $\psi(\mathbf{x}) = \text{constant}$ are the field lines (or stream lines) of $\mathbf{j}(\mathbf{x})$.*

Note that since $\psi(\mathbf{a}_k) = 0$ for $\mathbf{a}_k \in S_k$, the stream function on every surface is zero in at least one point. In general any constant may be added, which follows from (13).

Define the class of possible surface density functions stream functions on the surface S , denoted as $\Psi(S)$, as follows:

Definition 5. $\psi \in \Psi(S) \iff$

$$\begin{aligned} &\psi(\mathbf{x}) \text{ is constant on } C_l, \quad l = 1, \dots, L, \\ &\psi(\mathbf{x}) \text{ piecewise continuously differentiable on } S. \end{aligned} \tag{14}$$

We use the space of stream functions from $\Psi(S)$ as the representation of possible surface current densities.

5. Approximating the stream function by a conductor

In this section, a possible strategy of converting a given stream function into a conductor is discussed, showing the usefulness of the availability of a stream function. It is demonstrated by considering the surface current density on the surface $0 \leq x, y \leq 1, z = 0$, defined by the stream function

$$\psi(x, y) = \sin(\pi x) \sin(\pi y).$$

The current density is therefore

$$\mathbf{j}(x, y) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right) = (-\sin(\pi x) \cos(\pi y), \cos(\pi x) \sin(\pi y), 0).$$

First note that a current through a flat conductor laid on a surface can also be represented by a stream function; this stream function is constant outside the conductor since no current is flowing there. Fig. 3(a) plots the y -component of the current density on the line $y = 0.5$ (the x -components is zero on this line). The dotted line shows the current density of a conductor with 5 separate turns, each carrying the same current of $I_c := \frac{1}{5} = 0.2$ A. The current density is assumed to be constant inside the conductor.

More interesting is the graph where the stream functions are compared (Fig. 3(b)). The stream function of the conductor current can be chosen such that it closely follows the continuous stream function $\psi(x, y)$. Note that the conductor stream function is either constant, or linear with step size $\pm I_c$. We choose the center lines of the conductor (stream function 0.1, 0.3, 0.5, 0.7 and 0.9) such that they coincide with the

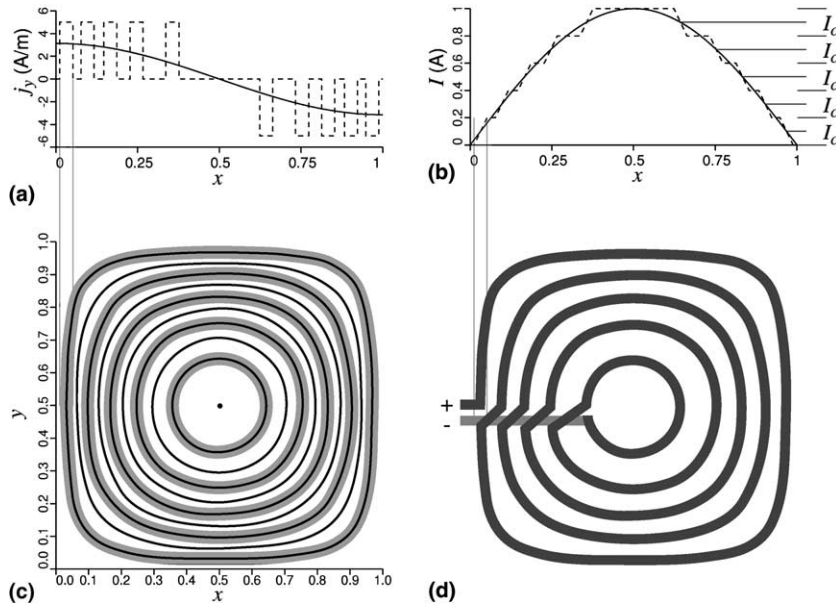


Fig. 3. Example of approximating a stream function by a conductor.

isolines of $\psi(x, y)$ the same value. The result is shown in Fig. 3(c). This process delivers unconnected conductors; a practical way of converting these into one conductor is shown in Fig. 3(d).

From this example, the following general strategy for a stream function given on one surface S can be derived:

1. Choose the ‘number of turns’ $N_{\text{turns}} \in \mathbb{N}^+$, and define the current I_c by

$$I_c := \frac{\max_{\mathbf{x} \in S} \psi(\mathbf{x}) - \min_{\mathbf{x} \in S} \psi(\mathbf{x})}{N_{\text{turns}}}.$$

2. The centerlines of the unconnected conductors are the isolines of $\psi(\mathbf{x})$ with step I_c , that is

$$\{\mathbf{x} \in S \mid \psi(\mathbf{x})\} = \min_{\mathbf{x} \in S} \psi(\mathbf{x}) + \left(n - \frac{1}{2}\right) I_c, \quad n = 1, \dots, N_{\text{turns}}.$$

Note that this set contains at least N_{turns} disjoint closed centerlines.

3. Form unconnected conductors from the centerlines by applying an (arbitrary) width. The width can be constrained by physical considerations, or may be subject to other optimality targets. For example, the width can be taken as large as possible to minimize the resistance and self-inductance, or as minimal as possible to reduce skin effects.
4. Convert the unconnected conductors into one conductor; a process as demonstrated in Fig. 3(d) can be used.

The background behind this strategy is indicated in Fig. 4. Consider the strip $\psi_1 \leq \psi(\mathbf{x}) \leq \psi_2$, with $\psi_2 - \psi_1$ taken small enough so that $\psi(\mathbf{x})$ is monotonous on this strip. On the line L through this strip as shown in the figure the current density is perpendicular, and the magnitude is $\psi'(s)$. Consider the quantity

$$\int_S \mathbf{j}(\mathbf{x}) f(\mathbf{x}) dS,$$

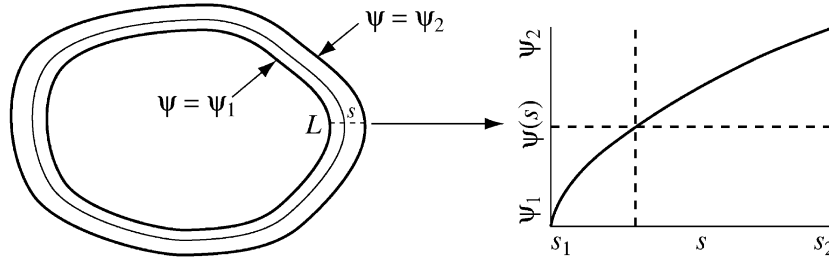


Fig. 4. Background of conversion strategy.

with $f(\mathbf{x})$ some arbitrary function, then the contribution from L is

$$\int_{s_1}^{s_2} \psi'(s) f(s) ds = \int_{\psi(s_1)}^{\psi(s_2)} f(\psi^{-1}(y)) dy \approx (\psi_2 - \psi_1) f\left(\psi^{-1}\left(\frac{\psi_1 + \psi_2}{2}\right)\right)$$

according the trapezoid integration rule, with absolute error $\frac{1}{12}(\psi_2 - \psi_1)^3 f''(\zeta)$.

This strategy has a lower order convergence if S consists of multiple unconnected surfaces, since the range of $\psi(\mathbf{x})$ over each surface is in general not a multiple of I_c . This condition can be imposed by additional constraints.

6. Numerical application of stream function

In this section, we show how the stream function may be applied to solve a shape optimization problem. The first step involves discretization of the stream function class $\Psi(S)$ (see Definition 5), that is write a stream function as

$$\psi(\mathbf{x}, t) = \sum_{n=1}^N s_n(t) \widehat{\psi}_n(\mathbf{x}), \tag{15}$$

where the coefficients $s_n(t)$ are the degrees of freedom, and $(\widehat{\psi}_n(\mathbf{x}))_{n=1}^N \in \Psi(S)$ is a given set of basis stream functions. The choice of basis functions is restricted: requirement (14) that states that $\psi(\mathbf{x}, t)$ is constant on each boundary must be *exactly* translatable into constraints on $s_n(t)$. If this is not the case, current will be ‘lost’ or ‘generated’ at the boundary, likely resulting in computed values for the magnetic field strength $\mathbf{H}(\mathbf{x})$ with large relative error.

A number of choices are possible for the basis functions. In this paper we work out some details for the basis functions associated with a mesh of polygons, where this mesh is in general a discretization of the surfaces. If the N nodes are denoted as $\xi_n, n = 1, \dots, N$ then basis function $\widehat{\psi}_n(\mathbf{x})$ is chosen to be 1 in ξ_n , 0 in all other nodes, and furthermore to be continuous everywhere, differentiable on each polygon, and linear on each vertex.

Let $\xi_{n_1}, \dots, \xi_{n_p}$ be the p nodes on a boundary, then requirement (14) stating that the stream function is constant on that boundary follows from the linearity of the basis functions on each vertex, and is given by

$$s_{n_1}(t) = \dots = s_{n_p}(t) \quad \text{for all } t.$$

This constraint is equivalent to replacing the basis functions $\widehat{\psi}_{n_1}(\mathbf{x}), \dots, \widehat{\psi}_{n_p}(\mathbf{x})$ and coefficients $s_{n_1}(t), \dots, s_{n_p}(t)$ by one basis function $\widehat{\psi}_{\text{bound}}(\mathbf{x}) := \sum_{j=1}^p \widehat{\psi}_{n_j}(\mathbf{x})$ with coefficient $s_{\text{bound}}(t)$, resulting in a reduction of the number of variables. Linear and quadratic functions in the variables $s_1(t), \dots, s_N(t)$ remain

linear in the reduced variables. Furthermore at least one node in every surface S_k must have a prescribed value; this reduces the number of variables with 1. Setting this prescribed value to zero is equivalent with omitting this variable.

We can therefore assume that the N basis functions $\hat{\psi}_1(\mathbf{x}), \dots, \hat{\psi}_N(\mathbf{x})$ are normalized, meaning that for every $s_1(t), \dots, s_N(t)$ the stream function $\psi(\mathbf{x}, t)$ as given by (15) is zero in at least one point in every S_k , and that (14) holds.

A number of quantities need to be computed from the basis functions $\hat{\psi}_n(\mathbf{x})$: the mutual resistance R_{mn} , given by (from (A.9))

$$R_{mn} = \int_S \frac{\hat{\mathbf{j}}_m(\mathbf{x}) \cdot \hat{\mathbf{j}}_n(\mathbf{x})}{\sigma(\mathbf{x})d(\mathbf{x})} dS \tag{16}$$

the mutual inductance (from (A.10)):

$$M_{mn} = \int_S \hat{\mathbf{A}}_m(\mathbf{x}) \cdot \hat{\mathbf{j}}_n(\mathbf{x}) dS = \int_S \hat{\mathbf{A}}_n(\mathbf{x}) \cdot \hat{\mathbf{j}}_m(\mathbf{x}) dS,$$

and the magnetic field $\mathbf{H}(\mathbf{x}, t)$.

Since R_{mn} involves integration of piece-wise continuous functions, it can be evaluated accurately using standard quadrature rules, such as Gauss–Legendre. The vector potential $\hat{\mathbf{A}}_m(\mathbf{x})$ occurring in the expression for M_{mn} is continuous everywhere, even on the surface S . However, usually for $\mathbf{x} \in S$ the computation of $\hat{\mathbf{A}}_m(\mathbf{x})$ is not trivial. For example, if the medium has constant magnetic permeability $\mu = \mu_0$, then

$$\hat{\mathbf{A}}_m(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int_S \frac{\hat{\mathbf{j}}_m(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} dS.$$

This types of integrals appear in commonly Boundary Element Methods. For methods to handle these type of integrals see for example [3].

Another issue is the treatment of the relation between source and induction currents, which is for a complete set of basic functions given by the differential equation (11). In our discretization, we apply this differential equation to the finite set of basis functions $(\hat{\psi}_n(\mathbf{x}))_{n=1}^N$.

Without loss of generality we can assume that the first N_s basis functions have their support of the source region, and the remaining $N_i := N - N_s$ basis functions on the induced region. Then denote the solution vector $[s_1(t), \dots, s_N(t)]^T$ and its partitioning in N_s and N_i elements as

$$\mathbf{s}(t) = \begin{bmatrix} \mathbf{s}_s(t) \\ \mathbf{s}_i(t) \end{bmatrix}.$$

Partition the mutual inductance matrix \mathbf{M} and resistance matrix \mathbf{R} analogously:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{si} \\ \mathbf{M}_{is} & \mathbf{M}_{ii} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{ss} & 0 \\ 0 & \mathbf{R}_{ii} \end{bmatrix}$$

(the off-diagonal matrices of \mathbf{R} are 0 because of (16) and the disjoint supports).

Then (11) is written as

$$\mathbf{M}_{is} \frac{ds_s}{dt} + \mathbf{M}_{ii} \frac{ds_i}{dt} + \mathbf{R}_{ii} \mathbf{s}_i = 0,$$

where \mathbf{M}_{is} , \mathbf{M}_{ii} and \mathbf{R}_{ii} are positive definite because \mathbf{M} and \mathbf{R} are positive definite.

The solution of the initial value problem, where $t \geq 0$ and $\mathbf{s}_i(0)$ is given, is

$$\mathbf{s}_i(t) = \mathbf{U}e^{-\mathbf{\Lambda}t}\mathbf{U}^{-1}\mathbf{s}_i(0) - \mathbf{U} \int_0^t e^{-\mathbf{\Lambda}(t-\tau)}\mathbf{U}^{-1}\mathbf{M}_{ii}^{-1}\mathbf{M}_{is} \frac{d\mathbf{s}_s(\tau)}{d\tau} d\tau, \quad t \geq 0, \quad (17)$$

where the matrices \mathbf{U} and $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_{N_i})$ are determined by the generalized eigenvalue problem

$$\mathbf{R}_{ii}\mathbf{U} = \mathbf{M}_{ii}\mathbf{U}\mathbf{\Lambda}.$$

Note that $\mathbf{\Lambda} \geq 0$, i.e. $\lambda_k \geq 0$, $k = 1, \dots, N_i$.

From (17) the physical interpretation of $\mathbf{\Lambda}$ and \mathbf{U} can be deduced: λ_i , $i = 1, \dots, N_i$ are the reciprocals of the *time constants* and the columns of \mathbf{U} are the corresponding *modes*.

Furthermore recall assumption (2), which stated that the source currents are driven by one source, that is

$$\mathbf{s}_s(t) = A(t)\tilde{\mathbf{s}}_s,$$

where $A(t)$ is a time-dependent function (the *amplitude*) and $\tilde{\mathbf{s}}_s$ is a vector that does not depend on time, and describes the static source current distribution. Note that $\tilde{\mathbf{s}}_s$ is the final solution we have to determine, since this vector fully defines the source currents.

If we define, for a given $A(t)$,

$$a_i(t) := \int_0^t A'(\tau) e^{-\lambda_i(t-\tau)} d\tau, \quad i = 1, \dots, N_i, \quad t \geq 0,$$

then (17) is equivalent to

$$\mathbf{s}_i(t) = \mathbf{U}e^{-\mathbf{\Lambda}t}\mathbf{U}^{-1}\mathbf{s}_i(0) - \mathbf{U} \text{diag}(a_1(t), \dots, a_{N_i}(t))\mathbf{U}^{-1}\mathbf{M}_{ii}^{-1}\mathbf{M}_{is}\tilde{\mathbf{s}}_s, \quad t \geq 0.$$

Therefore, for a fixed time t_0 and amplitude function $A(t)$, $\mathbf{s}_i(t_0)$ is linear in $\tilde{\mathbf{s}}_s$, and hence any property that depends linear on the induction currents also depends linear on the source currents described by the degrees of freedom $\tilde{\mathbf{s}}_s$.

Since in general $a_i(t) \approx 0$ for $\lambda_i t \gg 1$, typically only a few – depending on λ_i , $A(t)$ and t – eigenvalues and columns of \mathbf{U} need to be computed to evaluate (18) with sufficient accuracy.

A simplification occurs if $A(t)$ is the Heaviside function $H(t)$, defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t > 0. \end{cases}$$

For this case $a_i(t) = e^{-\lambda_i t}$, so (18) becomes

$$\mathbf{s}_i(t) = \mathbf{U}e^{-\mathbf{\Lambda}t}\mathbf{U}^{-1}(\mathbf{s}_i(0) - \mathbf{M}_{ii}^{-1}\mathbf{M}_{is}\tilde{\mathbf{s}}_s), \quad t > 0.$$

Furthermore,

$$\mathbf{s}_i(0^+) = \mathbf{s}_i(0) - \mathbf{M}_{ii}^{-1}\mathbf{M}_{is}\tilde{\mathbf{s}}_s, \quad (18)$$

so that for this special case the matrixes \mathbf{U} and $\mathbf{\Lambda}$ do not need to be computed.

7. Example: MRI gradient coils

In this section, the use of stream functions for shape optimization problems is demonstrated by the design of a gradient coil for an MRI system (for similar or other approaches, see for example [14,17]. Another example of the use of stream functions can be found in [2]).

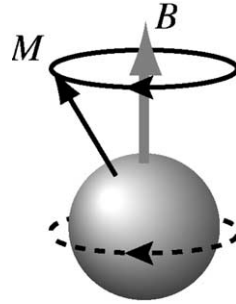


Fig. 5. Precession of magnetization vector around the magnetic field vector.

The purpose of an MRI system is to generate in vivo images of for example humans. It utilizes the *nuclear magnetic resonance* effect, which is a quantum mechanic phenomenon of some nuclei, such as the nucleus of hydrogen (proton). It has the macroscopic effect of a net volume magnetization $\mathbf{M}(\mathbf{x}, t)$, and appears when a magnetic field $\mathbf{B}(\mathbf{x}, t)$ is present. The magnitude of $\mathbf{M}(\mathbf{x}, t)$ increases and/or with the magnitude of $\mathbf{B}(\mathbf{x}, t)$ and density of the nuclei. The magnetization vector precesses around the direction of $\mathbf{B}(\mathbf{x}, t)$ (see Fig. 5), with a frequency which is proportional to the magnitude of $\mathbf{B}(\mathbf{x}, t)$. If the magnetization has a transversal component, an electromagnetic radiofrequency (RF) wave is emitted, which is received by an antenna.

Positional encoding is achieved by varying $\mathbf{B}(\mathbf{x}, t)$ both spatially and temporally, simultaneously receiving the signal. The frequency content of the received signal at each sample moment, combined with knowledge when the signal was received, allows for reconstruction of an image. See [18] for more information on this subject.

In MRI systems the temporal and spatial variation of the magnetic field is performed by the gradient coil. It usually consists of three independent electromagnetic coils, each driven by a controllable current source (known as gradient amplifier or gradient driver). Each coil is designed to deliver a substantially linear increasing field in a certain direction; the directions of the three coils are mutually orthogonal, and are denoted as x , y and z . To get sufficient magnetization, a magnet is also included, which delivers a uniform, constant and usually high magnetic field with flux density $\mathbf{B}_{\text{const}}$. The total magnetic field is therefore the superposition of the magnetic field of the gradient coil and the the magnet.

Note that we use the flux density B instead of the magnetic field H ; this is customary in MRI applications. However, because we assume the absence of magnetizable materials, both quantities are simply related by $\mathbf{B} = \mu_0 \mathbf{H}$, with $\mu_0 = 4\pi \times 10^{-7}$ [H/m] being the magnetic permeability of air.

In our example (see Fig. 6) we consider a gradient coil which has to fit in a metal, cylindrically shaped container. This container houses the (usually superconductive) magnet, and is represented by surface S_{inner} . Within the gradient coil the object to be imaged (patient) is positioned on a tabletop. We choose the gradient coil to consist of two concentric cylindrical surfaces S_{inner} and S_{outer} , which are coaxial with the container axes, and for aesthetical reasons of different length. The tabletop is flat, positioned parallel with, and at a certain distance below the axis of symmetry. Refer to Fig. 6 for the dimensions used in our example.

It is the objective to find the best conductor shape such that a linear increasing field in the upper direction (x) is generated. Since the background field $\mathbf{B}_{\text{const}}$ is much larger then the field of the gradient coil $\mathbf{B}_{\text{grad}}(\mathbf{x}, t)$, which is in our example directed in the z -direction, we are allowed to consider only the z -component of the magnetic field of the gradient coil, since $\|\mathbf{B}_{\text{grad}}(\mathbf{x}, t) + \mathbf{B}_{\text{const}}\| \approx B_{z,\text{grad}}(\mathbf{x}, t) + B_{z,\text{const}}$. Because of this property the magnetic field is linear in the current density.

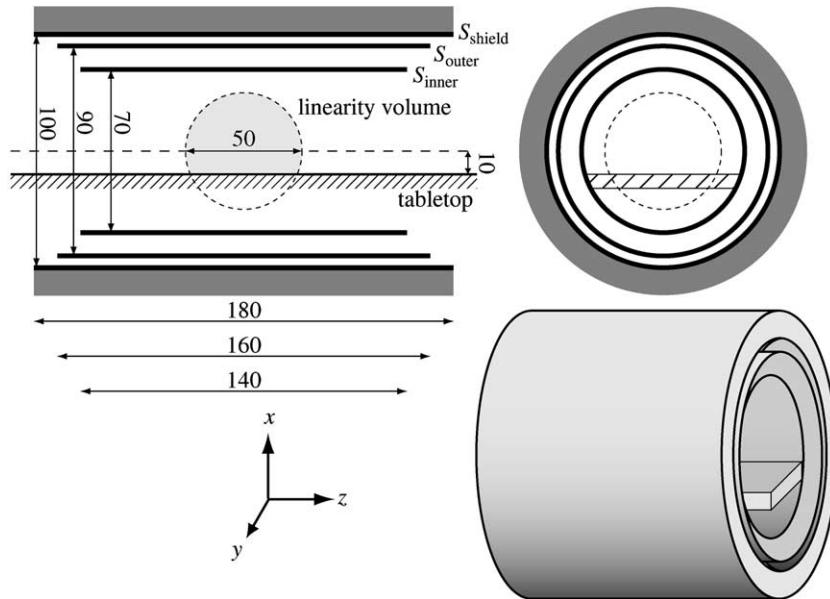


Fig. 6. Dimensions of the gradient coil in centimeters (side, front and perspective view).

We set constraints for the following properties:

(1) *Geometrical*: The source currents are on the surfaces S_{inner} and S_{outer} , the induction currents are on S_{shield} . Note that S_{shield} represents the magnetic container.

(2) *Magnetic field of source currents*: The magnetic field of the source currents must be substantially linear in the linearity volume (see Fig. 6). This is achieved by setting the following constraints:

- In the point $(x_{center}, 0, 0)$ the derivative $\partial B_z / \partial x$ is set to a prescribed value G , where the value of x_{center} has to be determined. We use $G = 10$ [mT/m].
- The target field is $B_z(x, y, z) = G \cdot (x - d)$; the linearity volume is

$$\{(x, y, z) | x^2 + y^2 + z^2 \leq R_{vol}^2, x \geq x_{table}\},$$

where in our example $R_{vol} = 25$ [cm], $x_{table} = -10$ [cm].

The constraints for the linearity are generated by choosing a sufficiently large number of control points at the boundary of the linearity volume, and requiring that the difference ΔB_z between the realized and target field in these points is not to exceed a certain maximum tol . This controls the image distortion in that point, since this is equal to $\Delta B_z / G \leq tol / G$. In our example the $N_\phi \times N_\theta$ control points and the tolerance in the points are generated by the following pseudo code, which sets the coordinates (x, y, z) and the tolerance tol for control point (i, j) , $i = 1, \dots, N_\phi$, $j = 1, \dots, N_\theta$:

```

 $\phi_i := 2\pi \cdot \frac{i-1}{N_\phi}$ ;
 $\theta_j := \pi \cdot \frac{j-1}{N_\theta}$ ;
 $tol := B_{tol}$ ;
 $x := R_{vol} \cos \phi_i \sin \theta_j$ ;
 $y := R_{vol} \sin \phi_i \sin \theta_j$ ;
 $z := R_{vol} \cos \theta_j$ ;
if  $x < x_{table}$  then

```

Control point below tabletop; shift to tabletop, adjust tolerance

$$\begin{aligned} y &= y \cdot \frac{x_{\text{table}}}{x}; \\ z &= z \cdot \frac{x_{\text{table}}}{x}; \\ x &= x_{\text{table}}; \\ \text{tol} &:= \text{tol} \cdot \left(\frac{\sqrt{x^2 + y^2 + z^2}}{R_{\text{vol}}} \right)^3; \end{aligned}$$

fi

We use $N_\phi = 12$, $N_\theta = 7$ and $B_{\text{tol}} = 120$ [μT], corresponding to maximally 12 mm image distortion at the boundary of the linearity volume.

(3) *Magnetic field of induction currents*: We consider induction currents as a result of an instantaneous switch of the source currents. The magnetic field generated by these currents causes image degradation, and we require this to be minimal. This is achieved by requiring that the absolute value of the z -component of the magnetic field generated by these induction currents at time $t = 0^+$ (see (18)) does not exceed a maximum value. We use the same control points as above, and a maximum value of 1 [μT].

The objective function is set to the magnetic energy of the magnetic field of the source currents. Minimizing this property is equivalent to requiring a minimum self-inductance. For this objective the least electric power is needed to ramp the magnetic field from zero to a certain value, therefore the gradient coil is optimized for fast switching of the magnetic field.

Fig. 7(a) shows the mesh of the three surfaces, which is a simple quadrilateral mesh. The total number of quadrilaterals used is 780, the number of nodes is 882.

After optimization we achieve a magnetic energy of 6.646 [J], and a dissipation (assuming $\sigma \cdot d = 8.5 \times 10^{-6}$ [Ω]), equivalent with 2 mm copper) of 2316 [W]. This energy is achieved for a value x_{center} of 3 [cm], which turns out to be the optimal value. The stream lines (contour lines of the stream function with step size 185 [A]) are shown in Fig. 7(b); these lines form the requested conductor. The current through this conductor required to generate the target derivative $G = (\partial B_z / \partial x)(x_{\text{center}}, 0, 0) = 10$ [mT/m] is 185 [A]. The surface S_{shield} is shown transparently. Fig. 8 shows the isolines with step size 0.1 [mT] of the z -component of the magnetic flux density in the x - y plane, $z = 0$. Note that the density of the field lines decreases below the table top.

The electric properties can be derived from these values. The resistance is $2316/185^2 = 67.67$ [m Ω], and the self-inductance is $2 \times 6.646/185^2 = 389$ [μH]. Linearly ramping the field from 0 to G in 1 [ms] requires a maximum of $2 \times 6.646/(185 \times 1 \times 10^{-3}) = 71.8$ [V]. Note that this values assume an optimal conductor conversion; in practice the necessary space for isolation between conductors will slightly increase the resistance and self-inductance.

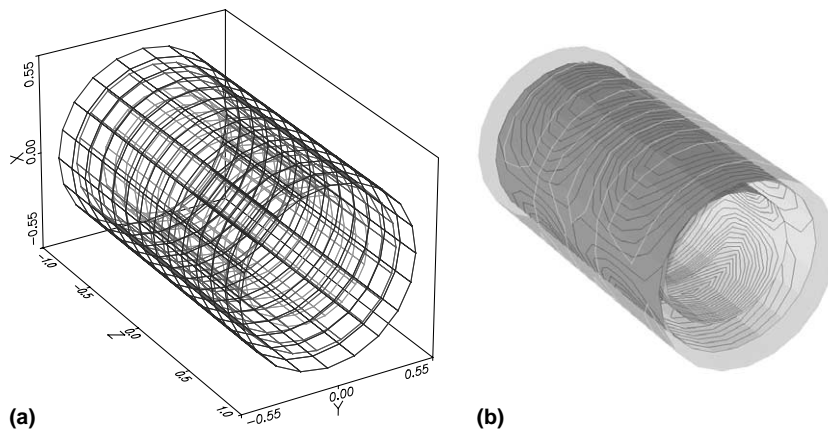


Fig. 7. The mesh and the computed stream lines of the example (185 A/line).

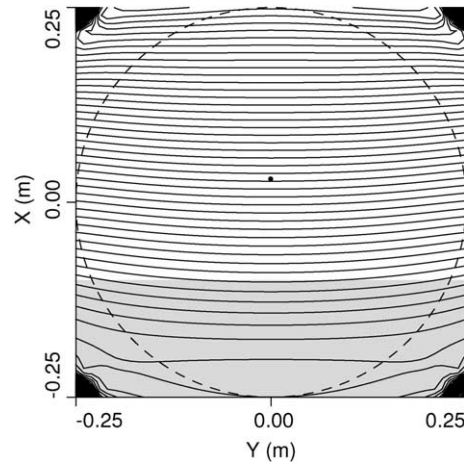


Fig. 8. Iso-field lines through the central plane.

In practice one may need to check the resulting field for compliance with the particular requirements. For example, usually a substantially better linearity is required close to the center, which may result in adding control points in that part of the region.

8. Conclusion

A formulation has been presented for quasi-static electromagnetic topological optimization problems involving good conductors only, where the required solution is a conductor shape, subject to geometrical and magnetic constraints. By broadening the class to current density distributions, and in addition assume that the geometry is thin, a scalar stream function is used as variable to represent the surface current density. From the stream function contour lines the conductor pattern is derived. This approach is very suited for applications where a linear or quadratic functional, such as the energy or dissipation, has to be optimized.

Acknowledgements

The author thanks W. Schilders and R. Mattheij, both at the University of Technology, Eindhoven, for their helpful assistance, and Philips Electronics for the facilities.

Appendix A. Derivation of the induction equation (11)

In this appendix relation (11) is proved. Consider a basis $(\hat{\mathbf{J}}_n(\mathbf{x}))_{n \in \mathbb{N}}$ in the separable Hilbert space of divergence-free vector functions defined on $V := V_{\text{source}} \cup V_{\text{ind}}$. Furthermore, since $V_{\text{source}} \cap V_{\text{ind}} = \emptyset$, we can assume without loss of generality that the support of every basis function $\hat{\mathbf{J}}_n(\mathbf{x})$ is either a subset of V_{source} or of V_{ind} , corresponding to subscripts $n \in \mathbb{N}_{\text{source}}$ and $n \in \mathbb{N}_{\text{ind}} := \mathbb{N} \setminus \mathbb{N}_{\text{source}}$ respectively.

We denote the magnetic and electric field generated by basis function $\hat{\mathbf{J}}_n(\mathbf{x})$ by $\hat{\mathbf{H}}_n(\mathbf{x})$ and $\hat{\mathbf{E}}_n(\mathbf{x})$ respectively, then because of the linearity of (6)

$$\begin{aligned}\mathbf{H}(\mathbf{x}, t) &= \sum_{n=1}^{\infty} I_n(t) \widehat{\mathbf{H}}_n(\mathbf{x}), \\ \mathbf{E}(\mathbf{x}, t) &= \sum_{n=1}^{\infty} I_n(t) \widehat{\mathbf{E}}_n(\mathbf{x}).\end{aligned}\tag{A.1}$$

Starting point is the law of conservation of energy, which follows from Maxwell's equations and the vector identity

$$\nabla \cdot (\mathbf{H} \times \mathbf{E}) = \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}).\tag{A.2}$$

When integrating over \mathbb{R}^3 , the left-hand side of (A.2) vanishes due to Gauss' law, and assuming that

$$\|\mathbf{H} \times \mathbf{E}\| = o(1/\|\mathbf{x}\|^2), \quad \|\mathbf{x}\| \rightarrow \infty.$$

Using (8) and (9) leads to

$$\underbrace{\frac{\partial}{\partial t} \frac{1}{2} \int_{\mathbb{R}^3} \mu \|\mathbf{H}(\mathbf{x}, t)\|^2 dV}_{E_{\text{magn}}} + \underbrace{\frac{\partial}{\partial t} \frac{1}{2} \int_{\mathbb{R}^3} \epsilon \|\mathbf{E}(\mathbf{x}, t)\|^2 dV}_{E_{\text{elec}}} + \underbrace{\int_{\mathbb{R}^3} \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{J}(\mathbf{x}, t) dV}_{P_{\text{diss}}} = 0.\tag{A.3}$$

The first two terms express the rate of change of the magnetic energy E_{magn} and electric energy E_{elec} respectively. The third term expresses the dissipation P_{diss} . At this point we assume that the second term is negligible, i.e.

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \epsilon \|\mathbf{E}(\mathbf{x}, t)\|^2 dV \ll \int_{\mathbb{R}^3} \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{J}(\mathbf{x}, t) dV.$$

Loosely stated this is equivalent to $\epsilon \|\frac{\partial \mathbf{E}}{\partial t}\| \ll \sigma \|\mathbf{E}\|$. Then Maxwell's equation (9) becomes

$$\nabla \times \mathbf{H} = \mathbf{J}.\tag{A.4}$$

This is known as the quasi-static case, since (7) and (A.4) fully specify the magnetic field strength $\mathbf{H}(\mathbf{x}, t)$ from the current density $\mathbf{J}(\mathbf{x}, t)$ without any time-derivative related equations. The electric field $\mathbf{E}(\mathbf{x}, t)$ is derived from $\mathbf{H}(\mathbf{x}, t)$ by the remaining equations.

Then, again using (A.2), we derive that

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \mu \mathbf{H}(\mathbf{x}, t) \cdot \widehat{\mathbf{H}}_n(\mathbf{x}) dV + \int_{\mathbb{R}^3} \mathbf{E}(\mathbf{x}, t) \cdot \widehat{\mathbf{J}}_n(\mathbf{x}) dV = 0.\tag{A.5}$$

Recall that for $\mathbf{x} \in V_{\text{ind}}$ relation (5) can be used, so that (A.5) evaluates to

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \mu \mathbf{H}(\mathbf{x}, t) \cdot \widehat{\mathbf{H}}_n(\mathbf{x}) dV + \int_{V_{\text{ind}}} \frac{\mathbf{J}(\mathbf{x}, t) \cdot \widehat{\mathbf{J}}_n(\mathbf{x})}{\sigma} dV = 0, \quad n \in \mathbb{N}_{\text{ind}}.\tag{A.6}$$

Substituting (A.1) leads to the (infinite) set of relations

$$\sum_{m=1}^{\infty} \left(\frac{dI_m(t)}{dt} \int_{\mathbb{R}^3} \mu \widehat{\mathbf{H}}_m(\mathbf{x}) \cdot \widehat{\mathbf{H}}_n(\mathbf{x}) dV + I_m(t) \int_{V_{\text{ind}}} \frac{\widehat{\mathbf{J}}_m(\mathbf{x}) \cdot \widehat{\mathbf{J}}_n(\mathbf{x})}{\sigma} dV \right) = 0, \quad n \in \mathbb{N}_{\text{ind}}.\tag{A.7}$$

This is (11), with the mutual inductance between basis function m and n given by

$$M_{mn} := \int_{\mathbb{R}^3} \mu \widehat{\mathbf{H}}_m(\mathbf{x}) \cdot \widehat{\mathbf{H}}_n(\mathbf{x}) dV, \quad m, n \in \mathbb{N},\tag{A.8}$$

and the mutual resistance by

$$R_{mn} := \int_{V_{\text{ind}}} \frac{\widehat{\mathbf{J}}_m(\mathbf{x}) \cdot \widehat{\mathbf{J}}_n(\mathbf{x})}{\sigma(\mathbf{x})} dV, \quad m, n \in \mathbb{N}_{\text{ind}}. \quad (\text{A.9})$$

Both M_{mn} and R_{mn} are symmetric. Since the magnetic energy is given by

$$E_{\text{magn}} = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_m(t) I_n(t) M_{mn},$$

and the dissipation in V_{ind} by

$$P_{\text{diss}}(V_{\text{ind}}) = \sum_{m \in \mathbb{N}_{\text{ind}}} \sum_{n \in \mathbb{N}_{\text{ind}}} I_m(t) I_n(t) R_{mn},$$

the matrices M_{mn} and R_{mn} are also positive definite.

Using the vector potential (defined by $\nabla \times \mathbf{A} = \mathbf{B}$, $\nabla \cdot \mathbf{A} = 0$), M_{mn} can also be written as

$$M_{mn} = \int_V \widehat{\mathbf{A}}_m(\mathbf{x}) \cdot \widehat{\mathbf{J}}_n(\mathbf{x}) dV = \int_V \widehat{\mathbf{J}}_m(\mathbf{x}) \cdot \widehat{\mathbf{A}}_n(\mathbf{x}) dV. \quad (\text{A.10})$$

This is computationally more advantageous, since it involves integration over the bounded volume V as opposed to the whole space as in (A.8).

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